THE SCHUR CONE AND THE CONE OF LOG CONCAVITY

DENNIS E. WHITE

ABSTRACT. Let $\{h_1, h_2, \dots\}$ be a set of algebraically independent variables. We ask which vectors are extreme in the cone generated by $h_i h_j - h_{i+1} h_{j-1}$ ($i \geq j > 0$) and h_i ($i > 0$). We call this cone the *cone of log concavity*. More generally, we ask which vectors are extreme in the cone generated by Schur functions of partitions with k or fewer parts. We give a conjecture for which vectors are extreme in the cone of log concavity. We prove the characterization in one direction and give partial results in the other direction.

1. Partitions and symmetric functions

Let $\{h_1, h_2, \dots\}$ be a set of algebraically independent variables. We ask which polynomials in these variables can be written as positive sums of products of polynomials of the form $h_i h_j - h_{i+1} h_{j-1}$ ($i \geq j > 0$) and h_i ($i > 0$). We have found it useful to place this question in the context of symmetric function, and, to that end, we need some preliminary definitions and results concerning partitions and symmetric functions. This material may be found in many other sources, most notably in [1] and in [4].

If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ with integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0$ and $N = \lambda_1 + \lambda_2 + \cdots + \lambda_m$, then λ is called a *partition of* N, and we write $\lambda \vdash N$. The integers λ_i are called the *parts* of the partition and $m = l(\lambda)$ is the *number of parts*. Another common notation for partitions is an exponential form. If the part k appears t_k times in the partition, we write k^{t_k}. Thus the partition of 18, $(4, 4, 2, 2, 2, 1, 1, 1, 1)$, can be written $4^2 2^3 1^4$.

Let $\mathcal{P}_N$ be the set of partitions of N, $\mathcal{P}_N^k$ be the set of partitions of N with k or fewer parts and $\mathcal{P}^k$ be the set of partitions with k or fewer parts. Let $p(N)$ be the size of $\mathcal{P}_N$.

A partition λ is sometimes called a *shape*, especially when it is described by a *Ferrers diagram*, an array of left-justified cells with λ_1 cells in the first row, λ_2 cells in the second row, etc.

If $\lambda \vdash N$ and $\mu \vdash N$, we say λ *dominates* μ if $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for all i and we write $\lambda \trianglerighteq \mu$ (and $\lambda \triangleright \mu$ if $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$). Dominance determines a partial order on $\mathcal{P}_N$.

If positive integers are placed in the cells of the shape λ, the resulting figure is called a *tableau*. The *content* of a tableau is a vector $\rho = (\rho_1, \rho_2, \dots)$ where ρ_i is the number of i's in the tableau. Vectors such as ρ are called *compositions*.

If the entries of the tableau weakly increase across rows and strictly decrease down columns, the tableau is called *semistandard*. The number of semistandard tableaux of shape λ and content ρ is $K_{\lambda,\rho}$, called the *Kostka number*. A well-known

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property of semistandard tableaux is that $K_{\lambda,\rho}$ does not depend upon the order of the entries in the vector ρ, so ρ is usually assumed to be a partition.

If T is a tableau of shape λ, then $w(T)$, called the *word of* T, is the word obtained by reading the entries in T from right to left across the first (top) row, then right to left across the second row, etc. If α is a subset of the entries appearing in T, then $w_\alpha(T)$ is the subword of $w(T)$ which uses just letters in α.

A word is a *lattice word* if, at any point in the word (reading left to right), the number of i's which have appeared is $\geq$ the number of $i+1$'s which have appeared.

The $\{h_1, h_2, \dots\}$ described above are usually defined to be the homogeneous symmetric functions in some set of indeterminates $x_1, x_2, \dots$. In this paper we will never need to refer to this underlying variable set. The fact that the h's are algebraically independent gives us the freedom to move around among symmetric function bases without regard to the underlying set of indeterminates.

We write $h_\rho = h_{\rho_1} h_{\rho_2} \dots$, where $\rho \vdash N$. The h_ρ, $\rho \vdash N$, form a basis of a vector space Λ^N of dimension $p(N)$.

We will use another basis, the Schur functions s_λ, extensively. We connect this basis with the h_ρ in two ways. The first is the equation

$$h_\rho = \sum_{\lambda \vdash N} K_{\lambda,\rho} s_\lambda \,,$$

where $\rho \vdash N$. The second is the Jacobi-Trudi identity,

$$s_\lambda = \det(h_{\lambda_i - i + j})_{1 \leq i,j \leq n}$$

where $n \geq l(\lambda)$.

If two Schur functions are multiplied together, the resulting symmetric function can be expanded as a linear combination of Schur functions. Suppose $(\rho^1, \rho^2, \dots, \rho^k)$ is a vector of partitions of $n_1, n_2, \dots, n_k$ respectively and $N = n_1 + n_2 + \dots + n_k$. Write

$$\prod_{i=1}^{k} s_{\rho^i} = \sum_{\lambda \vdash N} c^\lambda_{\rho^1,\dots,\rho^k} s_\lambda$$

The well-known Littlewood-Richardson rule states that the $c^\lambda_{\rho^1,\dots,\rho^k}$ are non-negative integers. (Symmetric functions which can be written in the Schur function basis with integer coefficients are called *Schur-integral*; symmetric functions which can be written in the Schur function basis with non-negative coefficients are called *Schur-positive*.) In fact, we can give a combinatorial description of these integers as follows.

Suppose $\tau = (\tau_1, \tau_2, \dots, \tau_N)$ and π is some permutation of $\{1, 2, \dots, N\}$. Let $\pi\tau = (\mu_1, \dots, \mu_N)$ where $\mu_{\pi_i} = \tau_i$. For example, if $\tau = (3, 2, 1, 1, 2, 2, 4, 2, 1)$ and $\pi = 4\,2\,3\,8\,1\,5\,9\,7\,6$, then $\pi\tau = (2, 2, 1, 3, 2, 1, 2, 1, 4)$.

Define a set partition of $\{1, 2, \dots, N\}$ corresponding to π, with blocks $B_1, \dots, B_k$, as follows.

$$B_1 = \{\pi_1, \pi_2, \dots, \pi_{n_1}\} \,,$$
$$B_2 = \{\pi_{n_1+1}, \pi_{n_1+2}, \dots, \pi_{n_1+n_2}\} \,,$$

and so on. For example, recalling that ρ^i is a partition of n_i, if $\rho^1 = (3, 2, 1, 1)$, $\rho^2 = (2, 2)$, and $\rho^3 = (4, 2, 1)$, and π is as above, then $B_1 = \{2, 3, 4, 8\}$, $B_2 = \{1, 5\}$ and $B_3 = \{6, 7, 9\}$.

Now let $\rho^1 \vee \rho^2 \vee \dots \vee \rho^k$ be the composition obtained by concatenating the partitions ρ^i. Continuing our example, we have $\rho^1 \vee \rho^2 \vee \rho^3 = (3, 2, 1, 1, 2, 2, 4, 2, 1)$.

Then, for any permutation π, $c^{\lambda}_{\rho^1,\dots,\rho^k}$ is the number of semistandard tableaux T of shape λ and content $\pi\rho^1 \vee \cdots \vee \rho^k$ such that $w_{B_i}(T)$ is a lattice word for each i.

In our example, $c^{\lambda}_{(3,2,1,1),(2,2),(4,2,1)}$ is the number of semistandard tableaux T of shape λ and content $(2,2,1,3,2,1,2,1,4)$ such that $w_{\{2,3,4,8\}}(T)$, $w_{\{1,5\}}(T)$ and $w_{\{6,7,9\}}(T)$ are each lattice words.

2. The cone of log-concavity

If A is a multiset from $\mathcal{P}^k$, define

$$w(A) = \sum_{\lambda \in A} |\lambda|$$

and

$$s_A = \prod_{\lambda \in A} s_\lambda \,.$$

The homogeneous degree of s_A (as a polynomial in the h's) is $w(A)$. Define

$$\mathcal{SP}^k_N = \{A \subseteq \mathcal{P}^k \mid w(A) = N\}\,.$$

The (N,k)-*Schur cone* is

$$\mathcal{C}^k_N = \Big\{ \sum_{A \in \mathcal{SP}^k_N} c_A s_A \mid c_A \geq 0 \Big\}$$

We say a function $s_A \in \mathcal{SP}^k_N$ is *extreme* in $\mathcal{C}^k_N$ if it cannot be written as a positive linear combination of other $s_A \in \mathcal{SP}^k_N$. We ask, for a given k, which elements $A \in \mathcal{SP}^k_N$ yield s_A which are extreme in this cone. We will frequently abuse notation by referring to the element A of $\mathcal{SP}^k_N$ instead of the corresponding product of Schur functions s_A.

We distinguish two obvious special cases. When $k = 1$, $s_A = h_\lambda$ where λ is the partition whose parts are the 1-row partitions of A. Since the h_λ form a basis of Λ^N and are the only vectors defining $\mathcal{C}^1_N$, they are the extreme vectors.

When $k \geq N$, then by the Littlewood-Richardson rule, the Schur functions s_λ are the extreme vectors.

It follows from the Jacobi-Trudi identity that the cone $\mathcal{C}^2_N$ consists of positive linear combinations of products of factors of the form

$$h_i h_j - h_{i+1}h_{j-1} \quad \text{and} \quad h_i \quad i \geq j \geq 1\,.$$

Thus, we call $\mathcal{C}^2_N$ the *cone of log concavity.*

There are many elements $A \in \mathcal{SP}^2_N$ which are not extreme in $\mathcal{C}^2_N$. For example,

$$s_{(3,1)}s_{(2)} = s_{(3,2)}s_{(1)} + s_{(1,1)}s_{(4)}\,.$$

In fact, the extreme set of $\mathcal{C}^2_6$ is just these 13 elements:

$s_{(6)}$	$s_{(4)}s_{(1,1)}$	$s_{(3)}s_{(2,1)}$
$s_{(5,1)}$	$s_{(3,1)}s_{(1,1)}$	${s_{(2,1)}}^2$
$s_{(4,2)}$	$s_{(2,2)}s_{(2)}$	$s_{(2)}{s_{(1,1)}}^2$
$s_{(3,3)}$	$s_{(2,2)}s_{(1,1)}$	${s_{(1,1)}}^3$
$s_{(3,2)}s_{(1)}$		

In this paper we conjecture a simple characterization of the extreme elements of $\mathcal{SP}^2_N$. We give a proof of this conjecture in one direction and we prove an important special case in the other direction.

3. The extreme set

The conjectured characterization of the extreme elements of $\mathcal{C}^2_N$ is the following.

Conjecture 1. *The collection of pairs $A \in \mathcal{SP}_N$ is in the extreme set of $\mathcal{C}^2_N$ if and only if no pair of partitions $\{\lambda, \mu\}$ in A satisfy any one of the following conditions:*

i. $\lambda = (\lambda_1 \geq \lambda_2 > 0)$, $\mu = (\mu_1 \geq \mu_2 > 0)$, *with*
$$\lambda_1 > \mu_1 \geq \lambda_2 > \mu_2\,;$$
ii. $\lambda = (\lambda_1 > \lambda_2 > 0)$, $\mu = (\mu_1 > 0)$, *with*
$$\lambda_1 \geq \mu_1 \geq \lambda_2\,;$$
iii. $\lambda = (\lambda_1 > 0)$, $\mu = (\mu_1 > 0)$.

If no pair of partitions in A satisfy any of these conditions, we say A is *nested*. The proof of one direction is easy.

Theorem 2. *If A is not nested then A is not in the extreme set of $\mathcal{C}^2_N$.*

Proof. Suppose a pair $\{\lambda, \mu\}$ satisfies the first condition. This implies $\lambda_1 \geq \mu_1 + 1$ and $\lambda_2 - 1 \geq \mu_2$. Therefore, by Jacobi-Trudi,

$$s_\lambda s_\mu = s_{(\lambda_1,\mu_2)} s_{(\mu_1,\lambda_2)} + s_{(\lambda_1,\mu_1+1)} s_{(\lambda_2-1,\mu_2)}\,. \tag{1}$$

Suppose a pair $\{\lambda, \mu\}$ satisfies the second condition. If $\lambda_1 > \mu_1$ then by Jacobi-Trudi,

$$s_\lambda s_\mu = s_{(\lambda_1)} s_{(\mu_1,\lambda_2)} + s_{(\lambda_2-1)} s_{(\lambda_1,\mu_1+1)}\,. \tag{2}$$

If $\mu_1 > \lambda_2$, by Jacobi-Trudi

$$s_\lambda s_\mu = s_{(\lambda_2)} s_{(\lambda_1,\mu_1)} + s_{(\lambda_1+1)} s_{(\mu_1-1,\lambda_2)}\,. \tag{3}$$

Finally, suppose a pair $\{\lambda, \mu\}$ satisfies the third condition. Then

$$s_\lambda s_\mu = s_{(\lambda_1,\mu_1)} + s_{(\lambda_1+1)} s_{(\mu_1-1)}\,. \tag{4}$$

□

Let $\mathcal{SSP}_N$ denote the nested sets $A \in \mathcal{SP}_N$. Thus, the extreme set of $\mathcal{C}^2_N$ is contained in $\mathcal{SSP}_N$.

For $A \in \mathcal{SP}_N$, let $\phi(A)$ be the partition defined by the parts of the partitions in A. For example, if $A = \{(4,2),(3,1),(3,2),(2)\}$, then $\phi(A) = 43^2 2^3 1$.

Several $A \in \mathcal{SP}_N$ might have the same $\phi(A)$. For $\lambda \vdash N$, let $\mathcal{SP}_\lambda = \{A \in \mathcal{SP}_N \mid \phi(A) = \lambda\}$. For example, if $\lambda = 42^2 1$, then $\{(4,2),(2,1)\}$ and $\{(4,1),(2,2)\}$ are both elements of $\mathcal{SP}_\lambda$. Similarly, define $\mathcal{SSP}_\lambda = \mathcal{SP}_\lambda \cap \mathcal{SSP}_N$. Note that for every λ, $\mathcal{SSP}_\lambda \neq \emptyset$.

Remark 3. *For $A \in \mathcal{SSP}_\lambda$, if λ has an even number of parts, then all the partitions of A have two parts, while if λ has an odd number of parts, then exactly one partition of A will have one part (and the remaining partitions in A will have two parts).*

There is an interesting connection between these nested sets of partitions and plane partitions. See [3] for the relevant definitions associated with plane partitions.

Proposition 4. *If $\lambda \vdash N$ has $2m$ parts, then there is a one-to-one correspondence ψ between elements of $\mathcal{SSP}_\lambda$ and plane partitions of N with shape (m, m) and parts λ_i.*

Proof. For each ρ in A, place ρ_1 in the first row of $\psi(A)$ and ρ_2 in the second row. Then write the rows in decreasing order. □

We illustrate this bijection with the following table, when $\lambda = 12^3 34^2 5$. There are four elements of $\mathcal{SSP}_\lambda$.

$A \in \mathcal{SSP}_\lambda$	$\psi(A)$
$\{(5,1),(4,2),(4,2),(3,2)\}$	$\begin{array}{cccc} 5 & 4 & 4 & 3 \\ 2 & 2 & 2 & 1 \end{array}$
$\{(5,1),(4,2),(4,3),(2,2)\}$	$\begin{array}{cccc} 5 & 4 & 4 & 2 \\ 3 & 2 & 2 & 1 \end{array}$
$\{(5,1),(4,4),(3,2),(2,2)\}$	$\begin{array}{cccc} 5 & 4 & 3 & 2 \\ 4 & 2 & 2 & 1 \end{array}$
$\{(5,3),(4,4),(2,1),(2,2)\}$	$\begin{array}{cccc} 5 & 4 & 2 & 2 \\ 4 & 3 & 2 & 1 \end{array}$

When the number of parts of λ is odd, the bijection ψ becomes an injection. For example, suppose $\lambda = 1^3 2^3 3 4^2$.

$A \in \mathcal{SSP}_\lambda$	$\psi(A)$
$\{(4,2),(4,2),(3,2),(1),(1,1)\}$	$\begin{array}{ccccc} 4 & 4 & 3 & 1 & 1 \\ 2 & 2 & 2 & 1 & 0 \end{array}$
$\{(4,2),(4,3),(2,2),(1),(1,1)\}$	$\begin{array}{ccccc} 4 & 4 & 2 & 1 & 1 \\ 3 & 2 & 2 & 1 & 0 \end{array}$
$\{(4,4),(3,2),(2,2),(1),(1,1)\}$	$\begin{array}{ccccc} 4 & 3 & 2 & 1 & 1 \\ 4 & 2 & 2 & 1 & 0 \end{array}$
$\{(4,4),(3),(2,1),(2,1),(2,1)\}$	$\begin{array}{ccccc} 4 & 3 & 2 & 2 & 2 \\ 4 & 1 & 1 & 1 & 0 \end{array}$
$\{(4,4),(3),(2,1),(2,2),(1,1)\}$	$\begin{array}{ccccc} 4 & 3 & 2 & 2 & 1 \\ 4 & 2 & 1 & 1 & 0 \end{array}$

To illustrate that the mapping to plane partitions is not a bijection, the plane partition

$$\begin{array}{ccccc} 4 & 4 & 3 & 2 & 1 \\ 2 & 2 & 1 & 1 & 0 \end{array}$$

does not appear in this list.

Note that when s_A is expanded in Schur functions, its support lies above $\phi(A)$ in dominance order, and its coefficients are non-negative. This is because these coefficients are exactly Littlewood-Richardson coefficients.

Proposition 5. *If $A \in \mathcal{SP}_N$, then*

$$s_A = \sum_{\mu \trianglerighteq \phi(A)} c_A^\mu s_\mu \, ,$$

with $c_A^{\phi(A)} = 1$ and $c_A^\mu \geq 0$.

Our primary tool in proving elements in $\mathcal{SSP}_N$ are extreme in $\mathcal{C}_N^2$ is the well-known Farkas' Lemma (see [2]). Farkas' Lemma states that a vector $\mathbf{v}$ is extreme in a cone if and only if there is a separating hyperplane, i.e., a hyperplane P such

that $\mathbf{v}$ lies on one side of P and all other generating vectors lie on the other side of P.

Since we are working in Λ^N and using the Schur functions as our basis, we determine separating hyperplanes by using the standard symmetric function inner product $\langle \cdot, \cdot \rangle$ for which the Schur functions are orthonormal.

Suppose $A \in \mathcal{SSP}_N$ and let f be a symmetric function such that $\langle f, s_B \rangle \leq 0$ for $B \in \mathcal{SSP}_N$, $B \neq A$, and $\langle f, s_A \rangle > 0$. Then we say f *separates* A.

Theorem 6. *There is a symmetric function f which separates A for $A \in \mathcal{SSP}_N$ if and only if A is extreme in $\mathcal{C}_N^2$.*

To prove Conjecture 1, we seek therefore a set of separating functions, one for each A. We now use Proposition 5 to reduce the amount of work we must do in finding separating functions.

Let $\lambda = \phi(A)$, $A \in \mathcal{SSP}_N$. We say the symmetric function f *separates A from above* if

i. $\langle f, s_A \rangle > 0$;
ii. $\langle f, s_B \rangle \leq 0$ for all B such that $\phi(B) \trianglerighteq \lambda$, $B \neq A$.

For example, for $N = 6$ and $A = \{(2,1),(2,1)\}$, we have $\lambda = 2^2 1^2$. If we take $f = s_{2^2 1^2} + s_{2^3} + s_{31^3} - s_{321}$, then f separates A from above.

Lemma 7. *If f separates A from above, then there is a symmetric function g which separates A.*

Proof. Let I be a dual order ideal in the dominance poset (see [3] for definitions) with $\lambda = \phi(A) \in I$. Let u be a symmetric function with the following properties:

$$\begin{aligned} &\langle u, s_A \rangle > 0\,; \\ &\langle u, s_B \rangle \leq 0 \quad \text{for all } B \neq A \text{ such that } \phi(B) \in I\,. \end{aligned} \tag{5}$$

We show how to grow I and u. Let μ be a partition which lies "just below" I, that is, $\mu \notin I$ and $J = I \cup \{\mu\}$ is a dual order ideal. Let

$$m = \max_{B \in \mathcal{SSP}_\mu} \{\langle u, s_B \rangle\}\,.$$

If $m \leq 0$, then u satisfies (5) for J. Otherwise, let $u^* = u - m s_\mu$. We show that u^* satisfies (5) for J.

For any B such that $\phi(B) \in I$, we have $\mu \not\trianglerighteq \phi(B)$, since $\mu \notin I$. Therefore, by Proposition 5, $\langle s_\mu, s_B \rangle = 0$, and we have

$$\langle u^*, s_B \rangle = \langle u, s_B \rangle \begin{cases} \leq 0 & \text{if } B \neq A \\ > 0 & \text{if } B = A \end{cases}.$$

For $B \in \mathcal{SSP}_\mu$, by Proposition 5, $\langle s_\mu, s_B \rangle = 1$, so we have

$$\langle u^*, s_B \rangle = \langle u, s_B \rangle - m \leq 0\,.$$

The proof now proceeds by iterating this construction. If f separates A from above, then f satisfies (5) for the dual order ideal generated by λ. By iterating the contruction above, we eventually arrive at a function g which satisfies (5) for I equal to the entire dominance poset. This is the same as saying g separates A. □

An important special case is the following corollary.

Corollary 8. *If $|\mathcal{SSP}_\lambda| = 1$, that is, $\mathcal{SSP}_\lambda = \{A\}$, then A is extreme in $\mathcal{C}_N^2$.*

Proof. The function s_λ separates A from above. $\square$

We can limit our search for separating functions even further by restricting to an interval in the dominance poset. Suppose $\phi(A) = \lambda$ and $\rho \trianglerighteq \lambda$. We will say the symmetric function f *separates* A on $[\lambda, \rho]$ if

i. $\langle f, s_\mu \rangle = 0$ whenever $\mu \notin [\lambda, \rho]$. That is, the *support of* f lies on $[\lambda, \rho]$;
ii. $\langle f, s_A \rangle > 0$;
iii. $\langle f, s_B \rangle \leq 0$ for all B such that $\phi(B) \in [\lambda, \rho]$, $B \neq A$.

Lemma 9. *If f separates A on $[\lambda, \rho]$, then f separates A from above.*

Proof. We show that for B such that $\phi(B) \triangleright \lambda$ but $\rho \ntrianglerighteq \phi(B)$, we have $\langle f, s_B \rangle = 0$. The support of s_B is $\trianglerighteq \phi(B)$ (Proposition 5). But then the support of s_B cannot be below ρ, so the support of s_B does not intersect the interval $[\lambda, \rho]$. $\square$

Again suppose $\phi(A) = \lambda$, where $A \in \mathcal{SSP}_N$. Three intervals above λ in dominance will be of particular interest to us. First, if $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m > 0)$, define

$$\lambda^+ = (\lambda_1 + 1, \lambda_2, \lambda_3, \ldots, \lambda_{m-1}, \lambda_m - 1)\,.$$

For example, if $\lambda = (4, 3, 3, 2, 2, 2, 1)$, then $\lambda^+ = (5, 3, 3, 2, 2, 2)$. We will find a symmetric function f which separates A on $[\lambda, \lambda^+]$ when λ has distinct parts. However, this interval is not sufficient when λ has repeated parts. For example, if $\lambda = 2^3 1^3$, then no such f separates $A = \{(2, 1), (2, 1), (2, 1)\}$ on this interval.

Now define

$$\lambda^{++} = \begin{cases} (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_k + 1, \lambda_{k+1} - 1, \ldots, \lambda_m - 1) & \text{if } m = 2k \\ (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_k + 1, \lambda_{k+1}, \lambda_{k+2} - 1, \ldots, \lambda_m - 1) & \text{if } m = 2k + 1\,. \end{cases}$$

In the previous example, $\lambda^{++} = (5, 4, 4, 2, 1, 1)$.

Conjecture 10. *For every $A \in \mathcal{SSP}_N$ with $\phi(A) = \lambda$ there is a symmetric function f such that f separates A on $[\lambda, \lambda^{++}]$.*

Clearly Conjecture 1 follows from Conjecture 10. Our evidence shows, however, that an even stronger conjecture may be true.

If $\lambda_{m-1} > \lambda_m$, define

$$\lambda^\dagger = \begin{cases} (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_{k-1} + 1, \lambda_k, \lambda_{k+1} - 1, \ldots, \lambda_{m-1} - 1, \lambda_m) & \text{if } m = 2k \\ (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_k + 1, \lambda_{k+1} - 1, \ldots, \lambda_{m-1} - 1, \lambda_m) & \text{if } m = 2k + 1\,. \end{cases}$$

Using the previous example, $\lambda^\dagger = (5, 4, 4, 1, 1, 1, 1)$.

For a partition $\mu = (\mu_1 \geq \mu_2 \geq \ldots \mu_m > 0)$ of N, define

$$\mu{\uparrow} = (\mu_1 + 1, \mu_2 + 1, \ldots, \mu_m + 1, 1)$$

and

$$\mu{\downarrow} = (\mu_1 - 1, \mu_2 - 1, \ldots, \mu_m - 1)\,.$$

Thus, $\mu{\uparrow}$ is a partition of $N + m + 1$ and $\mu{\downarrow}$ is a partition of $N - m$. Clearly, $\mu{\uparrow}{\downarrow} = \mu$ and if $\mu_{m-1} > \mu_m = 1$ then $\mu{\downarrow}{\uparrow} = \mu$. In the latter case, we say μ is $\downarrow$*-invertible.*

Now suppose $A \in \mathcal{SSP}_N$ with $\phi(A) = \lambda$. Define $A{\downarrow}$ by subtracting 1 from each part of each partition in A. For example, if $A = \{(5, 4), (3, 1), (2, 1), (1, 1)\}$, then $A{\downarrow} = \{(4, 3), (2), (1)\}$. Clearly, $\phi(A{\downarrow}) = \lambda{\downarrow}$.

We will say λ is *even* or *odd* according to whether λ has an even or odd number of parts.

Conjecture 11. *Suppose $A \in \mathcal{SSP}_N$ with $\phi(A) = \lambda$, λ even. If λ is $\downarrow$-invertible with $A\!\downarrow \in \mathcal{SSP}_{\lambda\downarrow}$, then there is a symmetric function f which separates A on $[\lambda, \lambda^\dagger]$. Otherwise, there is a symmetric function f which separates A on $[\lambda, \lambda^{++}]$.*

Clearly Conjecture 11 implies Conjecture 10 if λ is even.

Suppose λ is odd. Then $\lambda\!\uparrow$ is $\downarrow$-invertible. Define $A\!\uparrow$ to be the multiset of partitions obtained by adding 1 to each part of each partition in A, plus introducing a part of size 1 into the unique partition in A with one part. For example, if $A = \{(4), (2,1), (2,2)\}$, then $A\!\uparrow = \{(5,1), (3,2), (3,3)\}$. It is easy to check that $A\!\uparrow \in \mathcal{SSP}_{\lambda\uparrow}$.

The posets $[\lambda, \lambda^{++}]$ and $[\lambda\!\uparrow, \lambda\!\uparrow^\dagger]$ are easily seen to be isomorphic. Furthermore, every $B \in \mathcal{SSP}_\rho$, $\rho \in [\lambda, \lambda^{++}]$ corresponds to a $B\!\uparrow \in \mathcal{SSP}_{\rho\uparrow}$, $\rho\!\uparrow \in [\lambda\!\uparrow, \lambda\!\uparrow^\dagger]$ (although the reverse is not true).

Thus, if the coefficients of s_B agree with the coefficients of $s_{B\uparrow}$ on these intervals, then the separating symmetric function f for $A\!\uparrow$ will correspond to an appropriately defined separating symmetric function $f\!\downarrow$ for A.

Suppose $\rho \in [\lambda, \lambda^{++}]$. We need to show

$$\langle s_\rho, s_B \rangle = \langle s_{\rho\uparrow}, s_{B\uparrow} \rangle .$$

But this follows from the Littlewood-Richardson rule. The left hand side counts Littlewood-Richardson fillings of ρ using the parts from B while the right hand side counts the fillings of $\rho\!\uparrow$ using the parts from $B\!\uparrow$. There is a natural one-to-one corrspondence between these fillings by removing or adding the first column (which must contain the numbers $1, 2, \ldots, m, m+1$). $\square$

Theorem 12. *Conjecture 10 follows from Conjecture 11.*

The interval $[\lambda, \lambda^\dagger]$ (when defined) is not robust enough in general, even if λ is $\downarrow$-invertible.

For example, if $A = \{(3,1), (3,2), (2,2)\}$, with $\lambda = 3^2 2^3 1$, then there is no f separating A on $[\lambda, \lambda^\dagger] = [3^2 2^3 1, 4^2 2 1^3]$.

We have verified Conjecture 11 for $N \leq 20$.

4. Distinct partitions

We conclude this paper with a proof of the following theorem.

Theorem 13. *If λ has distinct parts and $A \in \mathcal{SSP}_\lambda$, then there is a symmetric function f which separates A on the interval $[\lambda, \lambda^+]$.*

Our tool for proving Theorem 13 is a natural iterative procedure which we describe next. Suppose $Y \subseteq \mathcal{SSP}_\lambda$. We say f *separates* Y *on* $[\lambda, \mu]$ if

i. f is Schur-integral.
ii. $\langle f, s_B \rangle \leq 0$ for all $B \in \mathcal{SSP}_\nu$, $\nu \in [\lambda, \mu]$, $\nu \neq \lambda$.
iii. $\langle f, s_B \rangle \leq 0$ for all $B \in \mathcal{SSP}_\lambda - Y$.
iv. $\langle f, s_A \rangle = k > 0$ for all $A \in Y$, where k does not depend on A.

Now suppose $X \subseteq Y \subseteq \mathcal{SSP}_\lambda$. We say g *partially separates* (X, Y) on $[\lambda, \mu]$ if

i. g is Schur-integral.
ii. $\langle g, s_B \rangle \leq 0$ for all $B \in \mathcal{SSP}_\nu$, $\nu \in [\lambda, \mu]$, $\nu \neq \lambda$.
iii. $\langle g, s_B \rangle \leq 0$ for all $B \in Y - X$.
iv. $\langle g, s_A \rangle = l > 0$ for all $A \in X$, where l does not depend on A.

Lemma 14. *Suppose f separates Y on $[\lambda, \mu]$ and g partially separates (X, Y) on $[\lambda, \mu]$. Then there exists an h which separates X on $[\lambda, \mu]$.*

Proof. Let

$$m = \max_{B \in \mathcal{SSP}_\lambda - Y} \langle g, s_B \rangle .$$

Pick a non-negative integer $b \geq m/k$. Let

$$h = g + bf - bks_\lambda .$$

We now verify that h has the required properties. Clearly, h is Schur-integral.

Now suppose $B \in \mathcal{SSP}_\nu$, $\nu \neq \lambda$. Then $\langle g, s_B \rangle \leq 0$, $\langle f, s_B \rangle \leq 0$, and $\langle s_\lambda, s_B \rangle = 0$ (by Proposition 5). Thus

$$\langle h, s_B \rangle \leq 0 ,$$

since $b \geq 0$.

Next, suppose $B \in \mathcal{SSP}_\lambda - Y$. Then $\langle g, s_B \rangle \leq m$, $\langle f, s_B \rangle \leq 0$, and $\langle s_\lambda, s_B \rangle = 1$ (again by Proposition 5). Then

$$\langle h, s_B \rangle \leq m - bk \leq 0 ,$$

since $b \geq 0$ and $b \geq m/k$.

Next, suppose $B \in Y - X$. Then $\langle g, s_B \rangle \leq 0$, $\langle f, s_B \rangle = k$, and $\langle s_\lambda, s_B \rangle = 1$. Thus

$$\langle h, s_B \rangle \leq bk - bk = 0 .$$

Finally, suppose $A \in X$. Then $\langle g, s_A \rangle = l$, $\langle f, s_A \rangle = k$, and $\langle s_\lambda, s_A \rangle = 1$, so

$$\langle h, s_A \rangle = l + bk - bk = l > 0 ,$$

and l is independent of the choice of A. □

We now apply Lemma 14 to the case where λ has distinct parts.

If λ is even ($m = 2k$), then the elements of $\mathcal{SSP}_\lambda$ are clearly counted by the Catalan numbers C_k. If λ is odd ($m = 2k - 1$) then the elements of $\mathcal{SSP}_\lambda$ are counted again by the Catalan numbers C_k. Furthermore, we may use the natural Catalan recursion induced by our realization of the partitions in $\mathcal{SSP}_\lambda$ as nestings. See Exercise 6.19, part (o) in [4].

Since the parts of λ are distinct, A is now a subset (not multiset) of 1- and 2-part partitions. If λ is even, then all the partitions in A are 2-part partitions. If λ is odd, then exactly one partition in A 1-part.

For A, $B \in \mathcal{SSP}_\lambda$, we say A and B *agree within* $\rho = (\rho_1, \rho_2)$ if whenever $\rho_1 > \mu_1 > \mu_2 > \rho_2$, then $\mu \in A$ if and only if $\mu \in B$.

Suppose $\rho_1 = \lambda_i$ and $\rho_2 = \lambda_j$. Define

$$\lambda[\rho] = (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \ldots, \lambda_{j-1}, \lambda_j - 1, \lambda_{j+1}, \ldots, \lambda_m) .$$

Note that $\lambda[\rho] \in [\lambda, \lambda^+]$.

Lemma 15. *Suppose A, $B \in \mathcal{SSP}_\lambda$, λ distinct. Suppose $\rho = (\lambda_i, \lambda_j)$, with $\rho \in A$, $\rho \notin B$, and A and B agree within ρ. The Littlewood-Richardson coefficients satisfy the following identity:*

$$c_A^{\lambda[\rho]} + 1 = c_B^{\lambda[\rho]} .$$

Furthermore, if $j = i + 1$, then $c_A^{\lambda[\rho]} = 0$ and $c_B^{\lambda[\rho]} = 1$.

Proof. We consider the set $\mathcal{K}$ all semistandard tableaux of shape $\lambda[\rho]$ and content λ. It is easy to describe these tableaux. Row 1 will have λ_1 1's in it, row 2 will have λ_2 2's in it, and so on until row $i-1$. Row i will have λ_i i's in it and one other number from $i+1, \ldots, j$. The first j rows will contain only the numbers from 1 to j, so row $j+1$ will have λ_{j+1} $j+1$'s in it, and so on to row m.

The coefficient $c_A^{\lambda[\rho]}$ counts those tableaux in $\mathcal{K}$ such that the pairs in A each form a lattice word in the standard reading order of the tableau.

Now consider a pair of tableau values, $r < s$, and the corresonding values of λ, $\mu = (\lambda_r, \lambda_s)$. If r and s lie outside the interval from i to j, then the entries in the tableau are uniquely determined as described above. This happens when $s \leq i$, when $r \geq j$, when $r \leq i$ and $s > j$, or when $r < i$ and $s \geq j$.

If these values lie within the interval from i to j (but not equal to i and j) then μ lies within ρ and will be in A if and only if it is in B.

Finally, if $r = i$ and $s = j$, then there is a unique tableau $T_0 \in \mathcal{K}$ in which the r-s word is not a lattice word. This tableau has a j at the end of row i. Then $c_B^{\lambda[\rho]}$ counts T_0 (since $\rho \notin B$), and $c_A^{\lambda[\rho]}$ does not.

If $j = i+1$, then T_0 is the only tableau in $\mathcal{K}$. □

Remark 16. *The number of tableaux in $\mathcal{K}$ is 2^{j-i-1}. These tableaux are in one-to-one correspondence with permutations of $j-i$ which avoid 321 and 312.*

We now complete the proof of Theorem 13.

Proof. Suppose $A \in \mathcal{SSP}_\lambda$. Order all the partitions in A with two parts from the "inside out." That is, list the partitions in A as $\rho^1, \rho^2, \ldots$, where all the partitions within ρ^j appear among $\rho^1, \rho^2, \ldots, \rho^{j-1}$.

Let

$$X_i = \{B \in \mathcal{SSP}_\lambda \mid B \text{ and } A \text{ agree inside } \rho_1, \rho_2, \ldots, \rho_m\}.$$

We then have this chain of subsets:

$$\mathcal{SSP}_\lambda = X_1 \supseteq X_2 \supseteq \cdots \supseteq X_m = \{A\}.$$

We wish to construct a chain of symmetric functions $(f_1, f_2, \ldots, f_m)$ such that each f_i separates X_i on $[\lambda, \lambda^+]$. Then f_m will be the desired separating vector for A.

Clearly we can start this chain with $f_1 = s_\lambda$, since s_λ separates $\mathcal{SSP}_\lambda$. To complete the chain, we use Lemma 14. We need a partially separating function at each stage. That is, with $X = X_{i+1}$ and $Y = X_i$, we need a function which partially separates (X, Y). Note that Y consists of all elements of $\mathcal{SSP}_\lambda$ which agree with A on $\{\rho_1, \ldots, \rho_i\}$ and $Y - X$ are those which do not agree with A on $\rho = \rho_{i+1}$. That is, $B \in Y$ means B and A agree inside ρ, but for such B, $B \in X$ if and only if $\rho \in B$. These are exactly the conditions needed to apply Lemma 15.

Let

$$g = (c_A^{\lambda[\rho]} + 1)s_\lambda - s_{\lambda[\rho]}.$$

We now verify that g partially separates (X, Y) on $[\lambda, \lambda^+]$, thus completing the proof.

Clearly g is Schur-integral.

For $B \in \mathcal{SSP}_\nu$, $\nu \neq \lambda$, $\langle s_\lambda, s_B \rangle = 0$ and $\langle s_{\lambda[\rho]}, s_B \rangle \geq 0$ (since s_B is Schur-positive). Thus $\langle g, s_B \rangle \leq 0$.

For $B \in Y - X$, $\langle s_\lambda, s_B \rangle = 1$ and $\langle s_{\lambda[\rho]}, s_B \rangle = c_B^{\lambda[\rho]}$. So by Lemma 15, $\langle g, s_B \rangle = c_A^{\lambda[\rho]} + 1 - c_B^{\lambda[\rho]} = 0$.

For $B \in X$, $\langle s_\lambda, s_B \rangle = 1$ and $\langle s_{\lambda[\rho]}, s_B \rangle = c_A^{\lambda[\rho]}$. Again by Lemma 15, $\langle g, s_B \rangle = c_A^{\lambda[\rho]} + 1 - c_A^{\lambda[\rho]} = 1$.

Therefore g partially separates (X, Y) as required. □

5. Remarks and acknowledgements

It is easy to describe the extreme vectors of $\mathcal{C}_N^{N-1}$. These vectors are all the Schur functions except s_{1^n}, which is replaced by $s_{1^{n-1}}s_1$. In a similar (but more complicated) fashion, it is possible to describe the extreme vectors of $\mathcal{C}_N^{N-2}$ and $\mathcal{C}_N^{N-3}$.

However, we have been unable to replace the conditions in Conjecture 1 with general conditions for the cone $\mathcal{C}_N^k$. Even the case $k = 3$ seems difficult, requiring that the syzygies in Equations (1) to (4) be replaced with appropriate syzygies for 3-row partitions.

Lemma 14 gives a general recipe for constructing separating vectors. Unfortunately, if λ has repeated parts, the chain of subsets of $\mathcal{SSP}_\lambda$ used in the proof of Theorem 13 and the required partially separating vectors seem much more elusive than in the case of distinct parts.

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(Dennis E. White) School of Mathematics, University of Minnesota, 127 Vincent Hall, 206 Church St SE, Minneapolis, MN 55455–0488

E-mail address: white@math.umn.edu

URL: http://www.math.umn.edu/~white